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Riemann–Hilbert problem for bi-orthogonal polynomials

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Abstract
The 3×3 matrix Riemann–Hilbert problem for bi-orthogonal polynomials with the third-degree polynomial potential functions is explicitly constructed. The developed approach can be extended to bi-orthogonal polynomials with arbitrary polynomial potentials.

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1. Introduction

The classical asymptotic theory of orthogonal polynomials [1, 2] extended to polynomials orthogonal with respect to the weight $\exp(-NV(z))$ [3–6] has gained essential progress after introduction of the Riemann–Hilbert (RH) problem approach [7] and the steepest descent method [8]. A particular implementation of these methods to the orthogonal polynomials on the real line can by found in [9, 10] while orthogonal polynomials on the circle are studied in [11].

Further extensions of the notion of orthogonal polynomials motivated by a number of applications to the random matrix theory, integrable systems, approximation theory and combinatorics include generalized orthogonal polynomials and bi-orthogonal polynomials. In the former case, the sequence of polynomials is orthogonal with respect to a sequence of measures [12, 13]

$$\int_{\mathbb{R}} P_n(\lambda)\,\mathrm{d}\rho_m(\lambda) = \delta_{nm}$$

while in the latter case, two sequences of polynomials are orthogonal to each other with respect to a two-dimensional measure [14–16],

$$\int_{\mathbb{R}}\int_{\mathbb{R}} P_n(\lambda)Q_m(\xi)\,\mathrm{d}\mu(\lambda,\xi) = \delta_{nm}.$$

In the simplest case, the two-dimensional measure has the form of the product $\mathrm{d}\mu(\lambda, \xi) = \exp(-V(\lambda) - W(\xi) + \lambda\xi)\,\mathrm{d}\lambda\,\mathrm{d}\xi$ where the polynomials $V(\lambda)$ and $W(\xi)$ are called *potentials*. Integration over ξ in the latter two-fold integral yields the sequence of measures, $\mathrm{d}\rho_m(\lambda) = \hat{\phi}_m(\lambda)\,\mathrm{d}\lambda = \int_{\mathbb{R}} Q_m(\xi)\,\mathrm{d}\mu(\lambda, \xi)$. The functions $\hat{\phi}_m(\lambda)$ here are called *dual functions* [15].

Generalized and bi-orthogonal polynomials are associated with a completely integrable system coming from *t*-deformations and Virasoro constraints and typically describing certain reductions of the 2-Toda lattice [12–16]. However, although the algebraic properties of the generalized and bi-orthogonal polynomials are extensively studied, knowledge of the asymptotic properties of such polynomials is limited. Basically, this is due to the absence of an adequate formulation of the relevant RH problem. Indeed, even though the 2×2 matrix RH problem formulation for the conventional orthogonal polynomials [7] admits a direct 2×2 extension to the case of generalized orthogonal polynomials [12], the relevant version for bi-orthogonal polynomials [14] exhibits its non-local nature.

Fortunately, recent study reveals the isomonodromy structure associated with bi-orthogonal polynomials and hence the principal possibility of formulating an $n \times n$ matrix RH problem which would have properties similar to those for the 2×2 matrix RH problem for conventional orthogonal polynomials [15]. In what follows, assuming the existence of bi-orthogonal polynomials for the third-degree polynomial potentials, we construct the relevant 3×3 matrix RH problem, as well as the 3×3 RH problem for the dual functions. In spite of its less physical importance, this case provides us the opportunity to develop the technique in the simplest non-trivial case. (The bi-orthogonal polynomials for the second-degree potentials are reduced to classical Hermite polynomials [14]. In some more involved cases such as $\deg V(\lambda) > 2$, $\deg W(\xi) = 2$, the bi-orthogonal polynomials can be expressed in terms of semi-classical orthogonal polynomials related to the 2×2 matrix RH problem studied in [7, 9–11] and other papers.

We stress that the method explained below can be extended to arbitrary polynomial potentials. We also point out a particular importance of the paper [17] which is useful for the construction and justification of the RH problem for the dual functions in the class of weights with rational log derivatives.

After the original version of the present paper was posted on the Internet [18], the Montreal group presented their methodology for constructing a similar RH problem [19]. Their idea for evaluation of the RH problem for the dual functions based on the study of path integrals concurs with ours but, in contrast to our cubic case, they consider arbitrary polynomial potentials. As for the RH problem for the original (wave) function, the authors of [19] rely on the so-called duality pairing, while our approach is based on the explicit integral representation of the wavefunction.

This paper is organized as follows. In section 2, we recall the matrix differential and difference equations satisfied by the bi-orthogonal polynomials and their dual functions. In section 3, we construct fundamental solutions for the differential–difference system for the dual functions which gives rise to the RH problem (equations (22)–(24)) for the dual functions. In section 4, we construct fundamental solutions of the differential–difference system for the bi-orthogonal polynomials and the relevant RH problem (equations (37)–(39)). In section 5, we discuss some properties and implications of the constructed RH problems.

2. Equations for the bi-orthogonal polynomials

Below, we consider monic polynomials $p_n(\lambda)$, $q_m(\xi)$ satisfying the orthogonality condition

$$\int_{\gamma_1} \mathrm{d}\lambda \int_{\gamma_2} \mathrm{d}\xi\, p_n(\lambda) q_m(\xi) \exp(-V(\lambda) - W(\xi) + t\lambda\xi) = h_n^2 \delta_{n,m} \tag{1}$$

where

$$V(\lambda) = \tfrac{1}{3}\lambda^3 + x\lambda \qquad W(\xi) = \tfrac{1}{3}\xi^3 + y\xi$$

$x, y, t \in \mathbb{C}, t \neq 0$; the contours $\gamma_i, i = 1, 2$, are the complex linear combinations of the elementary contours,

$$\gamma_i = \sum_{j=0}^{2} g_j^{(i)} \Gamma_j \qquad g_j^{(i)} \in \mathbb{C} \qquad i = 1, 2 \tag{2}$$

where each Γ_j is the sum of two rays

$$\Gamma_j = \left(\exp\left(\mathrm{i}\tfrac{2\pi}{3}(j-2)\right)\infty, 0\right] \cup \left[0, \exp\left(\mathrm{i}\tfrac{2\pi}{3}(j-1)\right)\infty\right) \qquad j = 0, 1, 2. \tag{3}$$

Because $\Gamma_0 + \Gamma_1 + \Gamma_2 = 0$, one of the parameters, $g_j^{(1)}$ (respectively $g_j^{(2)}$), can be put to zero and one of the nontrivial parameters, $g_j^{(i)}$, can be normalized to unity. Thus the set of contours γ_j and, therefore, the set of monic bi-orthogonal polynomials are parametrized by three constant complex parameters. (More general parametrization of the set of contours is introduced in [19]; however, in our cubic case, both kinds of parametrization are equivalent. We also note that the general cubic potentials can be reduced to the above form using the linear transformations in λ and ξ.)

We introduce the wavefunctions

$$\psi_n(\lambda) = \frac{1}{h_n} p_n(\lambda) \exp(-V(\lambda)) \qquad \phi_m(\xi) = \frac{1}{h_m} q_m(\xi) \exp(-W(\xi)) \tag{4}$$

and their Fourier–Laplace images $\hat{\psi}_n(\xi)$, $\hat{\phi}_m(\lambda)$ called the dual functions [15],

$$\hat{\psi}_n(\xi) = \int_{\gamma_1} \psi_n(\lambda) \exp(t\lambda\xi)\, \mathrm{d}\lambda \qquad \hat{\phi}_m(\lambda) = \int_{\gamma_2} \phi_m(\xi) \exp(t\lambda\xi)\, \mathrm{d}\xi. \tag{5}$$

The orthogonality condition (1) now reads

$$\int_{\gamma_1} \mathrm{d}\lambda \int_{\gamma_2} \mathrm{d}\xi\, \psi_n(\lambda)\phi_m(\xi) \exp(t\lambda\xi) = \int_{\gamma_1} \psi_n(\lambda)\hat{\phi}_m(\lambda)\, \mathrm{d}\lambda = \int_{\gamma_2} \hat{\psi}_n(\xi)\phi_m(\xi)\, \mathrm{d}\xi = \delta_{nm}. \tag{6}$$

It implies certain relations between the introduced functions (4) and (5). We refer to [15] for the general case and present the final result for our particular situation here:

$$\begin{aligned}
&\lambda\psi_n(\lambda) = \sum_{m=n-2}^{n+1} a_{n,m}\psi_m(\lambda) \qquad \partial_\lambda\psi_n(\lambda) = -t\sum_{m=n-1}^{n+2} b_{n,m}\psi_m(\lambda) \\
&\partial_x\psi_n(\lambda) = \sum_{m=n}^{n+1} u_{n,m}\psi_m(\lambda) \qquad \partial_y\psi_n(\lambda) = -\sum_{m=n-1}^{n} v_{n,m}\psi_m(\lambda) \\
&\partial_t\psi_n(\lambda) = w_n\psi_n(\lambda) - \sum_{m=n-3}^{n-1} A_{n,m}\psi_m(\lambda)
\end{aligned} \tag{7}$$

$$\begin{aligned}
&\xi\phi_m(\xi) = \sum_{n=m-2}^{m+1} b_{n,m}\phi_n(\xi) \qquad \partial_\xi\phi_m(\xi) = -t\sum_{n=m-1}^{m+2} a_{n,m}\phi_n(\xi) \\
&\partial_y\phi_m(\xi) = \sum_{n=m}^{m+1} v_{n,m}\phi_n(\xi) \qquad \partial_x\phi_m(\xi) = -\sum_{n=m-1}^{m} u_{n,m}\phi_n(\xi) \\
&\partial_t\phi_m(\xi) = w_m\phi_m(\xi) - \sum_{n=m-3}^{m-1} A_{n,m}\phi_n(\xi)
\end{aligned} \tag{8}$$

where the coefficients $a_{n,m}$, $b_{n,m}$, $u_{n,m}$, $v_{n,m}$, $A_{n,m}$ are described in more detail in [18].

Using definition (5) and the equations given above, it is straightforward that the dual functions satisfy equations

$$\begin{aligned} &\partial_\xi \hat{\psi}_n(\xi) = \sum_{m=n-2}^{n+1} t a_{n,m} \hat{\psi}_m(\xi) && \xi \hat{\psi}_n(\xi) = \sum_{m=n-1}^{n+2} b_{n,m} \hat{\psi}_m(\xi) \\ &\partial_x \hat{\psi}_n(\xi) = \sum_{m=n}^{n+1} u_{n,m} \hat{\psi}_m(\xi) && \partial_y \hat{\psi}_n(\xi) = -\sum_{m=n-1}^{n} v_{n,m} \hat{\psi}_m(\xi) \\ &\partial_t \hat{\psi}_n(\xi) = -w_n \hat{\psi}_n(\xi) + \sum_{m=n+1}^{n+3} A_{n,m} \hat{\psi}_m(\xi) \end{aligned} \tag{9}$$

$$\begin{aligned} &\partial_\lambda \hat{\phi}_m(\lambda) = \sum_{n=m-2}^{m+1} t b_{n,m} \hat{\phi}_n(\lambda) && \lambda \hat{\phi}_m(\lambda) = \sum_{n=m-1}^{m+2} a_{n,m} \hat{\phi}_n(\lambda) \\ &\partial_y \hat{\phi}_m(\lambda) = \sum_{n=m}^{m+1} v_{n,m} \hat{\phi}_n(\lambda) && \partial_x \hat{\phi}_m(\lambda) = -\sum_{n=m-1}^{m} u_{n,m} \hat{\phi}_n(\lambda) \\ &\partial_t \hat{\phi}_m(\lambda) = -w_m \hat{\phi}_m(\lambda) + \sum_{n=m+1}^{m+3} A_{n,m} \hat{\phi}_n(\lambda). \end{aligned} \tag{10}$$

Using the above equations, 3-vectors $\Psi_n(\lambda) = (\psi_n(\lambda), \psi_{n-1}(\lambda), \psi_{n-2}(\lambda))^T$, $\Phi_m(\xi) = (\phi_m(\xi), \phi_{m-1}(\xi), \phi_{m-2}(\xi))^T$, and dual vectors $\hat{\psi}_n(\xi) = (\hat{\psi}_n(\xi), \hat{\psi}_{n-1}(\xi), \hat{\psi}_{n-2}(\xi))^T$, $\hat{\phi}_m(\lambda) = (\hat{\phi}_m(\lambda), \hat{\phi}_{m-1}(\lambda), \hat{\phi}_{m-2}(\lambda))^T$ satisfy the following systems of difference and differential equations with 3×3 matrix coefficients [15],

$$\begin{aligned} &\Psi_{n+1}(\lambda) = R_n(\lambda)\Psi_n(\lambda) \qquad \frac{\partial \Psi_n}{\partial \lambda}(\lambda) = A_n(\lambda)\Psi_n(\lambda) \\ &\frac{\partial \Psi_n}{\partial x} = U_n(\lambda)\Psi_n \qquad \frac{\partial \Psi_n}{\partial y} = V_n(\lambda)\Psi_n \qquad \frac{\partial \Psi_n}{\partial t} = W_n(\lambda)\Psi_n \end{aligned} \tag{11}$$

$$\begin{aligned} &\Phi_{m+1}(\xi) = Q_m(\xi)\Phi_m(\xi) \qquad \frac{\partial \Phi_m}{\partial \xi}(\xi) = B_m(\xi)\Phi_m(\xi) \\ &\frac{\partial \Phi_m}{\partial x} = \mathcal{U}_m \Phi_m \qquad \frac{\partial \Phi_m}{\partial y} = \mathcal{V}_m \Phi_m \qquad \frac{\partial \Phi_m}{\partial t} = \mathcal{W}_m \Phi_m \end{aligned} \tag{12}$$

$$\begin{aligned} &\hat{\Psi}_{n+1}(\xi) = \hat{R}_n(\xi)\hat{\Psi}_n(\xi) \qquad \frac{\partial \hat{\Psi}_n}{\partial \xi}(\xi) = \hat{A}_n(\xi)\hat{\Psi}_n(\xi) \\ &\frac{\partial \hat{\Psi}_n}{\partial x}(\xi) = \hat{U}_n(\xi)\hat{\Psi}_n(\xi) \qquad \frac{\partial \hat{\Psi}_n}{\partial y}(\xi) = \hat{V}_n(\xi)\hat{\Psi}_n(\xi) \qquad \frac{\partial \hat{\Psi}_n}{\partial t}(\xi) = \hat{W}_n(\xi)\hat{\Psi}_n(\xi) \end{aligned} \tag{13}$$

$$\begin{aligned} &\hat{\Phi}_{m+1}(\lambda) = \hat{Q}_m(\lambda)\hat{\Phi}_m(\lambda) \qquad \frac{\partial \hat{\Phi}_m}{\partial \lambda}(\lambda) = \hat{B}_m(\lambda)\hat{\Phi}_m(\lambda) \\ &\frac{\partial \hat{\Phi}_m}{\partial x}(\lambda) = \hat{\mathcal{U}}_m(\lambda)\hat{\Phi}_m(\lambda) \qquad \frac{\partial \hat{\Phi}_m}{\partial y}(\lambda) = \hat{\mathcal{V}}_m(\lambda)\hat{\Phi}_m(\lambda) \qquad \frac{\partial \hat{\Phi}_m}{\partial t}(\lambda) = \hat{\mathcal{W}}_m(\lambda)\hat{\Phi}_m(\lambda) \end{aligned} \tag{14}$$

where the expressions for the 3×3 matrix coefficients are given in [18].

The compatibility conditions of the above equations yield a reduction of the 2-Toda lattice [12]. On the other hand, this nonlinear system describes the isomonodromy deformations with respect to the parameters $x, y, t \in \mathbb{C}$ and $n, m \in \mathbb{N}$ of the 3×3 matrix differential equations in λ and ξ. Below we show that by expanding the underlying idea of [7] it is possible to construct fundamental solutions of systems (11)–(14) in terms of the (unknown) bi-orthogonal polynomials using merely the fact of the existence of linear systems (11)–(14) rather than the systems themselves.

3. Particular solutions of the matrix equations and the Riemann–Hilbert problem for the dual functions

Because of the above-mentioned independence of the relevant monodromy data from the deformation parameter t, without loss of generality, we restrict ourselves to $t > 0$. This assumption allows us to simplify our calculations while the final result will be valid for arbitrary $t \in \mathbb{C}\backslash\{0\}$. Below, we give the detailed derivation of the RH problem for the dual functions $\hat{\psi}_n(\xi)$, see (5), satisfying (9). Similar details for $\hat{\phi}_m(\lambda)$ satisfying (10) can be found in [18] or easily reconstructed from those for $\hat{\psi}_n(\xi)$ using substitutions $\psi \leftrightarrow \phi$, $\lambda \leftrightarrow \xi, n \leftrightarrow m, x \leftrightarrow y, p \leftrightarrow q, \gamma_1 \leftrightarrow \gamma_2$ and $g_j^{(1)} \leftrightarrow g_j^{(2)}$.

Let us introduce the auxiliary functions

$$\hat{\psi}_n^{(j)}(\xi) = \int_{\Gamma_j} \psi_n(\lambda)\exp(t\lambda\xi)\,\mathrm{d}\lambda \tag{15}$$

where the contours Γ_j, $j = 0, 1, 2$, are defined in (3). Due to (5) and (2), the dual and auxiliary functions are related by

$$\hat{\psi}_n(\xi) = \sum_{j=0}^{2} g_j^{(1)}\hat{\psi}_n^{(j)}(\xi) \tag{16}$$

and satisfy the same equations (9). Taking into account the orthogonality condition (6), we find that the functions $F_n(\xi)$,

$$F_n(\xi) = \frac{\exp(W(\xi))}{2\pi\mathrm{i}}\int_{\gamma_2}\frac{\hat{\Psi}_n(\zeta)\exp(-W(\zeta))}{\zeta-\xi}\,\mathrm{d}\zeta \tag{17}$$

for $n \geqslant 2$ also satisfy (9).

Observing that $\hat{\psi}_n^{(j)}$, $j = 1, 2$, are combinations of the *independent* Airy functions and their derivatives

$$\hat{\psi}_n^{(j)}(\xi) = \frac{1}{h_n}p_n(\partial_\tau)\int_{\Gamma_j}\exp\left(-\tfrac{1}{3}\lambda^3+\tau\lambda\right)\mathrm{d}\lambda\bigg|_{\tau=t\xi-x} \tag{18}$$

we construct the fundamental piece-wise holomorphic solution of the system of the 3×3 matrix equations (13) and (14),

$$\hat{\Psi}_n(\xi) = \begin{pmatrix} \hat{\psi}_n^{(1)}(\xi) & \hat{\psi}_n^{(2)}(\xi) & F_n(\xi) \\ \hat{\psi}_{n-1}^{(1)}(\xi) & \hat{\psi}_{n-1}^{(2)}(\xi) & F_{n-1}(\xi) \\ \hat{\psi}_{n-2}^{(1)}(\xi) & \hat{\psi}_{n-2}^{(2)}(\xi) & F_{n-2}(\xi)\end{pmatrix} \qquad n \geqslant 4. \tag{19}$$

The jump property of the Cauchy integral yields the relations

$$F_n^+(\xi) - F_n^-(\xi) = \left(g_j^{(2)} - g_{j+1}^{(2)}\right)\hat{\psi}_n(\xi) \qquad \arg\xi = \frac{2\pi}{3}(j-1) \qquad j = 0, 1, 2 \tag{20}$$

where $g_3^{(i)} \equiv g_0^{(i)}$. Thus the matrix function $\hat{\Psi}_n(\xi)$ has the following jumps across the rays

$\ell_j = \left\{\xi \in \mathbb{C}\colon \arg\xi = \frac{2\pi}{3}(j-1)\right\}$, $j = 0, 1, 2$, oriented towards infinity

$$\hat{\Psi}_n^+(\xi) = \hat{\Psi}_n^-(\xi)\begin{pmatrix}1 & 0 & \left(g_j^{(2)} - g_{j+1}^{(2)}\right)\left(g_1^{(1)} - g_0^{(1)}\right)\\ 0 & 1 & \left(g_j^{(2)} - g_{j+1}^{(2)}\right)\left(g_2^{(1)} - g_0^{(1)}\right)\\ 0 & 0 & 1\end{pmatrix} \qquad \xi \in \ell_j \qquad j = 0, 1, 2. \tag{21}$$

Using the well-known asymptotics of the Airy integrals in the complex domain,

$$\int_{\Gamma_0} \exp\left(-\tfrac{1}{3}\lambda^3 + \tau\lambda\right) \mathrm{d}\lambda = -\mathrm{i}\sqrt{\pi}\tau^{-1/4}\exp\left(-\tfrac{2}{3}\tau^{3/2}\right)\left(1 + \mathrm{O}(\tau^{-3/2})\right) \qquad \arg\tau \in \left(-\tfrac{2\pi}{3}, \tfrac{2\pi}{3}\right)$$

$$\int_{\Gamma_1} \exp\left(-\tfrac{1}{3}\lambda^3 + \tau\lambda\right) \mathrm{d}\lambda = \mathrm{i}\sqrt{\pi}\tau^{-1/4}\exp\left(-\tfrac{2}{3}\tau^{3/2}\right)\left(1 + \mathrm{O}(\tau^{-3/2})\right) \qquad \arg\tau \in \left(\tfrac{2\pi}{3}, 2\pi\right)$$

$$\int_{\Gamma_2} \exp\left(-\tfrac{1}{3}\lambda^3 + \tau\lambda\right) \mathrm{d}\lambda = -\sqrt{\pi}\tau^{-1/4}\exp\left(\tfrac{2}{3}\tau^{3/2}\right)\left(1 + \mathrm{O}(\tau^{-3/2})\right) \qquad \arg\tau \in \left(0, \tfrac{4\pi}{3}\right)$$

the asymptotics of the Cauchy integrals for (17),

$$F_n(\xi) = -\frac{h_n}{2\pi\mathrm{i}}\xi^{-n-1}\exp(W(\xi))(1 + \mathrm{O}(\xi^{-1}))$$

which is obtained by applying the orthogonality condition (6), we construct *the RH problem for the dual functions* $\hat{\psi}_n(\xi)$.

Riemann–Hilbert problem 1. *Find a piece-wise holomorphic* 3×3 *matrix function* $\hat{\Psi}_n^{\mathrm{RH}}(\xi)$ *with the following properties:*

1. $\hat{G}^{-1}(\xi)\hat{\Psi}_n^{\mathrm{RH}}(\xi)\exp(-\hat{\Theta}(\xi)) \to I$ *as* $\xi \to \infty$ *where*

$$\hat{G}(\xi) = \begin{pmatrix} \frac{\sqrt{\pi}}{h_n}(t\xi)^{\frac{1}{4}} & \frac{\sqrt{\pi}}{h_n}(-1)^n(t\xi)^{\frac{1}{4}} & -\frac{h_n}{2\pi\mathrm{i}}\xi^{-2}\\ \frac{\sqrt{\pi}}{h_{n-1}}(t\xi)^{-\frac{1}{4}} & \frac{\sqrt{\pi}}{h_{n-1}}(-1)^{n-1}(t\xi)^{-\frac{1}{4}} & -\frac{h_{n-1}}{2\pi\mathrm{i}}\xi^{-1}\\ \frac{\sqrt{\pi}}{h_{n-2}}(t\xi)^{-\frac{3}{4}} & \frac{\sqrt{\pi}}{h_{n-2}}(-1)^{n-2}(t\xi)^{-\frac{3}{4}} & -\frac{h_{n-2}}{2\pi\mathrm{i}} \end{pmatrix} \tag{22}$$

$$\hat{\Theta}(\xi) = \mathrm{diag}\left(\tfrac{2}{3}(t\xi)^{3/2} - x(t\xi)^{1/2} + \tfrac{1}{2}(n-1)\ln(t\xi)\right.$$
$$\left. -\tfrac{2}{3}(t\xi)^{3/2} + x(t\xi)^{1/2} + \tfrac{1}{2}(n-1)\ln(t\xi),\ \tfrac{1}{3}\xi^3 + y\xi - (n-1)\ln\xi\right);$$

2. *Across the rays* $\arg\xi = \frac{2\pi}{3}(j-1)$, $j = 1, 2, 3$, *oriented towards infinity,* $\hat{\Psi}_n^{\mathrm{RH}}(\xi)$ *has the jumps*

$$\hat{\Psi}_n^{\mathrm{RH}+}(\xi) = \hat{\Psi}_n^{\mathrm{RH}-}(\xi)\hat{S}_j \qquad \arg\xi = \tfrac{2\pi}{3}(j-1) \tag{23}$$

where plus and minus indicate the limiting values of $\Psi_n^{\mathrm{RH}}(\xi)$ *on the jump contour from the left and from the right, respectively, and*

$$\hat{S}_1 = \begin{pmatrix}1 & 0 & \left(g_1^{(2)} - g_2^{(2)}\right)\left(g_1^{(1)} - g_2^{(1)}\right)\\ -\mathrm{i} & 1 & \mathrm{i}\left(g_1^{(2)} - g_2^{(2)}\right)\left(g_2^{(1)} - g_0^{(1)}\right)\\ 0 & 0 & 1\end{pmatrix}$$

$$\hat{S}_2 = \begin{pmatrix}1 & -\mathrm{i} & \left(g_2^{(2)} - g_0^{(2)}\right)\left(g_1^{(1)} - g_2^{(1)}\right)\\ 0 & 1 & \mathrm{i}\left(g_2^{(2)} - g_0^{(2)}\right)\left(g_1^{(1)} - g_0^{(1)}\right)\\ 0 & 0 & 1\end{pmatrix}$$

$$\hat{S}_3 = \begin{pmatrix}1 & 0 & \left(g_0^{(2)} - g_1^{(2)}\right)\left(g_0^{(1)} - g_2^{(1)}\right)\\ -\mathrm{i} & 1 & \mathrm{i}\left(g_0^{(2)} - g_1^{(2)}\right)\left(g_1^{(1)} - g_0^{(1)}\right)\\ 0 & 0 & 1\end{pmatrix}.$$

Across the ray $\arg\xi = -\frac{\pi}{3}$ *oriented towards infinity, the jump condition holds*

$$\hat{\Psi}_n^{\mathrm{RH}+}(\xi) = \hat{\Psi}_n^{\mathrm{RH}-}(\xi)\hat{\Sigma} \qquad \arg\xi = -\frac{\pi}{3} \qquad \hat{\Sigma} = \begin{pmatrix} 0 & \mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{24}$$

The dual functions $\hat{\psi}_n(\xi)$, $\hat{\psi}_{n-1}(\xi)$ and $\hat{\psi}_{n-2}(\xi)$ form the 3-vector $\hat{\Psi}_n(\xi)$ related to the solution of RH problem 1 by the following equations:

$$\hat{\Psi}_n(\xi) = \begin{cases} \hat{\Psi}_n^{\mathrm{RH}}(\xi)R^{(1)} & \arg\xi \in \left(-\frac{\pi}{3}, 0\right) \\ \hat{\Psi}_n^{\mathrm{RH}}(\xi)\hat{S}_1^{-1}R^{(1)} & \arg\xi \in \left(0, \frac{2\pi}{3}\right) \\ \hat{\Psi}_n^{\mathrm{RH}}(\xi)\hat{S}_2^{-1}\hat{S}_1^{-1}R^{(1)} & \arg\xi \in \left(\frac{2\pi}{3}, \frac{4\pi}{3}\right) \\ \hat{\Psi}_n^{\mathrm{RH}}(\xi)\hat{S}_3^{-1}\hat{S}_2^{-1}\hat{S}_1^{-1}R^{(1)} & \arg\xi \in \left(\frac{4\pi}{3}, \frac{5\pi}{3}\right) \end{cases} \tag{25}$$

where the 3-vector $R^{(1)} = \left(g_1^{(1)} - g_2^{(1)}, \mathrm{i}\left(g_2^{(1)} - g_0^{(1)}\right), 0\right)^T$.

Remark 1. In the cubic case the general construction formulated in [19], yields mainly the same RH problem modulo notations. In our terms, the authors of [19] study the matrix function

$$\begin{pmatrix} 0 & 0 & 1 \\ m_{11} & m_{12} & 0 \\ m_{21} & m_{22} & 0 \end{pmatrix} \hat{\Phi}_{n+1}^T \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

where m_{ij} are properly chosen constants. Under this modification, the jump condition for $\hat{\Phi}_{n+1}$ across the rays ℓ_j, which can be obtained from (21), turns into a jump condition (3.1.6) of [19] across the lines L_μ. Equation (3.2.20) of [19] describes the Stokes phenomenon for the Fourier–Laplace transforms $\hat{\phi}_m$ which, in our cubic case, are linear combinations of the Airy functions and their derivatives. It is interesting that, in the cubic case, the Stokes phenomenon takes place for one half of the directions $\mathcal{R}_k$ described in the general construction of [19], namely, the jumps across $\arg\xi = \frac{\pi}{3} + \frac{2\pi}{3}(j-1)$ are trivial. Furthermore, since the remaining rays $\mathcal{R}_k$ coincide with the rays L_μ, the respective jump matrices, separately described in [19], are replaced in (23) by their products $\hat{S}_j$. The most significant difference between both formulations is jump condition (24) which is absent in [19]. In our approach, it comes from the RH problem for the Airy functions, see, e.g. [20], i.e. from the fact that the formal Airy asymptotics $z^{-1/4}\exp\left(\frac{2}{3}z^{3/2}\right)$, $z^{-1/4}\exp\left(-\frac{2}{3}z^{3/2}\right)$ are single-valued on the complex z-plane cut along $[0, \infty)$. In our approach, the cyclic relation

$$\hat{S}_1\hat{S}_2\hat{S}_3\hat{\Sigma} = I \tag{26}$$

whose validity can be checked by straightforward computation, ensures the continuity of the RH problem at the origin and reflects the fact that all solutions of an ODE with polynomial coefficients are entire functions. The absence of a similar jump in the construction of [19] means that, implicitly, the authors of the mentioned paper formulate their RH problem on a Riemann surface (four-sheeted in the cubic case) rather than on the plane.

4. Particular solutions of the matrix equations and the Riemann–Hilbert problem for the wavefunctions

The method of construction of the Riemann–Hilbert problem for the wavefunctions $\psi_n(\lambda)$ and $\phi_m(\xi)$ is more involved because of the less elementary structure of their integral representations in comparison to those for dual functions. Below, we give the detailed derivation of the RH problem for the $\psi_n(\lambda)$. Similar details for $\phi_m(\xi)$ can be found in [18] or obtained from those for $\psi_n(\lambda)$ using substitutions $\psi \leftrightarrow \phi, \lambda \leftrightarrow \xi, n \leftrightarrow m, x \leftrightarrow y$, and $g_j^{(1)} \leftrightarrow g_j^{(2)}$.

Let $\tilde{\ell}_0^{(j)}$ be an oriented contour connecting a finite point ξ_0 with infinity within the sector of the exponential decay of the function $\hat{\psi}_n^{(j)}(\xi)$ (18). Namely, let $\tilde{\ell}_0^{(j)}$ be asymptotic to the ray

$$\tilde{\ell}_0^{(j)} \sim \left[0, \exp\left(-\mathrm{i}\tfrac{2\pi}{3} j\right)\infty\right). \tag{27}$$

Let $\tilde{\Gamma}_j$ be an infinite oriented contour asymptotic to the rays

$$\tilde{\Gamma}_j \sim \left(\exp\left(\mathrm{i}\tfrac{2\pi}{3}\left(j-\tfrac{3}{2}\right)\right)\infty, 0\right] \cup \left[0, \exp\left(\mathrm{i}\tfrac{2\pi}{3}\left(j-\tfrac{1}{2}\right)\right)\infty\right) \qquad j = 0, 1, 2 \tag{28}$$

located within the sector (28) and intersecting the ray

$$\ell_j = \left[0, \exp\left(\mathrm{i}\tfrac{2\pi}{3}(j-1)\right)\infty\right) \tag{29}$$

so that $\tilde{\Gamma}_j \cap \ell_j = \{\xi_j\}$.

The 'bricks' which we use to build up the appropriate integral representations are the 'inverse' Fourier–Laplace transforms

$$\tilde{\psi}_n^{(j)}(\lambda) = \frac{t}{2\pi\mathrm{i}} \int_{\tilde{\ell}_0^{(j)}} \exp(-t\lambda\xi)\hat{\psi}_n^{(j)}(\xi)\,\mathrm{d}\xi \qquad \tilde{F}_n(\lambda) = \frac{t}{2\pi\mathrm{i}} \int_{\tilde{\Gamma}_j} \exp(-t\lambda\xi) F_n(\xi)\,\mathrm{d}\xi. \tag{30}$$

Functions (30) satisfy the differential equations in λ, x and y in (7) but not the recursion relation and the differential equation in t because of the appearance of inappropriate off-integral terms resulting from integration by parts. In more detail:

$$\lambda\tilde{\psi}_n^{(k)}(\lambda) = \tfrac{1}{2\pi\mathrm{i}} \exp(-t\lambda\xi_0)\hat{\psi}_n^{(k)}(\xi_0) + \text{appropriate terms} \tag{31}$$

$$\begin{aligned}\lambda\tilde{F}_n(\lambda) &= \tfrac{1}{2\pi\mathrm{i}} \exp(-t\lambda\xi_j)\left(F_n^{+}(\xi_j) - F_n^{-}(\xi_j)\right) + \text{appropriate terms}\\ &= \tfrac{1}{2\pi\mathrm{i}} \exp(-t\lambda\xi_j)\left(g_j^{(2)} - g_{j+1}^{(2)}\right)\sum_{k=0}^{2} g_k^{(1)}\hat{\psi}_n^{(k)}(\xi_j) + \text{appropriate terms.}\end{aligned} \tag{32}$$

In the second line of (32), we have used jump condition (20) and definition (16). Combining (31) at $\xi_0 = \xi_j$ and (32), it is possible to eliminate the off-integral terms and find such a combination $\tilde{F}_n^{(j)}(\lambda)$ of $\tilde{\psi}_n^{(k)}(\lambda)$ and $\tilde{F}_n(\lambda)$ which satisfies the system (7),

$$\tilde{F}_n^{(j)}(\lambda) = \tilde{F}_n(\lambda) - \left(g_j^{(2)} - g_{j+1}^{(2)}\right)\sum_{k=0}^{2} g_k^{(1)}\tilde{\psi}_n^{(k)}(\lambda). \tag{33}$$

Identity $\sum_j \hat{\psi}^{(j)}(\xi) \equiv 0$ entails that $\sum_j \tilde{\psi}^{(j)}(\lambda)$ also satisfies (7). Thus it is possible to eliminate one of the values $\tilde{\psi}_n^{(k)}(\lambda)$ from (33). For technical reasons, we prefer to use the latter opportunity and introduce the following solutions of (7):

$$\begin{aligned}
\tilde{F}_n^{(0)}(\lambda) &= \tilde{F}_n(\lambda) - \left(g_0^{(2)} - g_1^{(2)}\right)\left(g_0^{(1)} - g_1^{(1)}\right)\tilde{\psi}_n^{(0)}(\lambda) - \left(g_0^{(2)} - g_1^{(2)}\right)\left(g_2^{(1)} - g_1^{(1)}\right)\tilde{\psi}_n^{(2)}(\lambda)\\
\tilde{F}_n^{(1)}(\lambda) &= \tilde{F}_n(\lambda) - \left(g_1^{(2)} - g_2^{(2)}\right)\left(g_1^{(1)} - g_0^{(1)}\right)\tilde{\psi}_n^{(1)}(\lambda) - \left(g_1^{(2)} - g_2^{(2)}\right)\left(g_2^{(1)} - g_0^{(1)}\right)\tilde{\psi}_n^{(2)}(\lambda)\\
\tilde{F}_n^{(2)}(\lambda) &= \tilde{F}_n(\lambda) - \left(g_2^{(2)} - g_0^{(2)}\right)\left(g_0^{(1)} - g_2^{(1)}\right)\tilde{\psi}_n^{(0)}(\lambda) - \left(g_2^{(2)} - g_0^{(2)}\right)\left(g_1^{(1)} - g_2^{(1)}\right)\tilde{\psi}_n^{(1)}(\lambda).
\end{aligned}$$

Define 3×3 matrix functions

$$\Psi_n(\lambda) = \begin{pmatrix} \psi_n(\lambda) & \tilde{F}_n^{(0)}(\lambda) & \tilde{F}_n^{(1)}(\lambda)\\ \psi_{n-1}(\lambda) & \tilde{F}_{n-1}^{(0)}(\lambda) & \tilde{F}_{n-1}^{(1)}(\lambda)\\ \psi_{n-2}(\lambda) & \tilde{F}_{n-2}^{(0)}(\lambda) & \tilde{F}_{n-2}^{(1)}(\lambda) \end{pmatrix} \qquad n \geqslant 4. \tag{34}$$

The asymptotics at infinity of $\psi_n(\lambda)$ is elementary

$$\psi_n(\lambda) = \frac{\lambda^n}{h_n} \exp\left(-\tfrac{1}{3}\lambda^3 - x\lambda\right)(1 + \mathrm{O}(\lambda^{-1})). \tag{35}$$

The asymptotics of $\tilde{F}_n^{(j)}(\lambda)$ can be found using the conventional steepest descent method

$$\begin{aligned}
\tilde{F}_n^{(1)}(\lambda) &= \frac{\mathrm{i}th_n}{4\pi^{3/2}}(t\lambda)^{-\frac{n+1}{2}-\frac{1}{4}} \exp\left(-\tfrac{2}{3}(t\lambda)^{3/2} + y(t\lambda)^{1/2}\right)(1 + \mathrm{O}(\lambda^{-1/2})) + \left(g_1^{(2)} - g_2^{(2)}\right)\\
&\quad \times \left(g_1^{(1)} - g_0^{(1)}\right)\frac{\lambda^n}{h_n} \exp\left(-\tfrac{1}{3}\lambda^3 - x\lambda\right)(1 + \mathrm{O}(\lambda^{-1})) \qquad \arg\lambda \in \left(-\tfrac{2\pi}{3}, 0\right)\\
\tilde{F}_n^{(1)}(\lambda) &= \frac{\mathrm{i}th_n}{4\pi^{3/2}}(t\lambda)^{-\frac{n+1}{2}-\frac{1}{4}} \exp\left(-\tfrac{2}{3}(t\lambda)^{3/2} + y(t\lambda)^{1/2}\right)(1 + \mathrm{O}(\lambda^{-1/2})) + \left(g_1^{(2)} - g_2^{(2)}\right)\\
&\quad \times \left(g_2^{(1)} - g_0^{(1)}\right)\frac{\lambda^n}{h_n} \exp\left(-\tfrac{1}{3}\lambda^3 - x\lambda\right)(1 + \mathrm{O}(\lambda^{-1})) \qquad \arg\lambda \in \left(0, \tfrac{2\pi}{3}\right)\\
\tilde{F}_n^{(0)}(\lambda) &= -\frac{th_n}{4\pi^{3/2}}(-1)^n(t\lambda)^{-\frac{n+1}{2}-\frac{1}{4}} \exp\left(\tfrac{2}{3}(t\lambda)^{3/2} - y(t\lambda)^{1/2}\right)(1 + \mathrm{O}(\lambda^{-1/2})) + \left(g_0^{(2)} - g_1^{(2)}\right)\\
&\quad \times \left(g_2^{(1)} - g_1^{(1)}\right)\frac{\lambda^n}{h_n} \exp\left(-\tfrac{1}{3}\lambda^3 - x\lambda\right)(1 + \mathrm{O}(\lambda^{-1})) \qquad \arg\lambda \in \left(0, \tfrac{2\pi}{3}\right)\\
\tilde{F}_n^{(0)}(\lambda) &= -\frac{th_n}{4\pi^{3/2}}(-1)^n(t\lambda)^{-\frac{n+1}{2}-\frac{1}{4}} \exp\left(\tfrac{2}{3}(t\lambda)^{3/2} - y(t\lambda)^{1/2}\right)(1 + \mathrm{O}(\lambda^{-1/2})) + \left(g_0^{(2)} - g_1^{(2)}\right)\\
&\quad \times \left(g_0^{(1)} - g_1^{(1)}\right)\frac{\lambda^n}{h_n} \exp\left(-\tfrac{1}{3}\lambda^3 - x\lambda\right)(1 + \mathrm{O}(\lambda^{-1})) \qquad \arg\lambda \in \left(\tfrac{2\pi}{3}, \tfrac{4\pi}{3}\right)\\
\tilde{F}_n^{(2)}(\lambda) &= -\frac{\mathrm{i}th_n}{4\pi^{3/2}}(t\lambda)^{-\frac{n+1}{2}-\frac{1}{4}} \exp\left(-\tfrac{2}{3}(t\lambda)^{3/2} + y(t\lambda)^{1/2}\right)(1 + \mathrm{O}(\lambda^{-1/2})) + \left(g_2^{(2)} - g_0^{(2)}\right)\\
&\quad \times \left(g_0^{(1)} - g_2^{(1)}\right)\frac{\lambda^n}{h_n} \exp\left(-\tfrac{1}{3}\lambda^3 - x\lambda\right)(1 + \mathrm{O}(\lambda^{-1})) \qquad \arg\lambda \in \left(\tfrac{2\pi}{3}, \tfrac{4\pi}{3}\right)\\
\tilde{F}_n^{(2)}(\lambda) &= -\frac{\mathrm{i}th_n}{2\pi^{3/2}}(t\lambda)^{-\frac{n+1}{2}-\frac{1}{4}} \exp\left(-\tfrac{2}{3}(t\lambda)^{3/2} + y(t\lambda)^{1/2}\right)(1 + \mathrm{O}(\lambda^{-1/2})) + \left(g_2^{(2)} - g_0^{(2)}\right)\\
&\quad \times \left(g_1^{(1)} - g_2^{(1)}\right)\frac{\lambda^n}{h_n} \exp\left(-\tfrac{1}{3}\lambda^3 - x\lambda\right)(1 + \mathrm{O}(\lambda^{-1})) \qquad \arg\lambda \in \left(\tfrac{4\pi}{3}, 2\pi\right).
\end{aligned}$$

Using the above asymptotics and the linear constraint for $\tilde{F}_n^{(j)}$,

$$\begin{aligned}
&\tilde{F}_n^{(0)}(\lambda) + \tilde{F}_n^{(1)}(\lambda) + \tilde{F}_n^{(2)}(\lambda) = g_F\psi_n(\lambda)\\
&\qquad g_F = g_0^{(2)}\left(g_2^{(1)} - g_1^{(1)}\right) + g_1^{(2)}\left(g_1^{(1)} - g_0^{(1)}\right) + g_2^{(2)}\left(g_0^{(1)} - g_2^{(1)}\right)
\end{aligned} \tag{36}$$

we find the RH problem for our bi-orthogonal polynomials.

Riemann–Hilbert problem 2. *Find a piece-wise holomorphic* 3×3 *matrix function* $\Psi_n^{\mathrm{RH}}(\lambda)$ *with the following properties:*

1. $G^{-1}(\lambda)\Psi_n^{\mathrm{RH}}(\lambda)\exp(-\Theta(\lambda)) \to I$ *as* $\lambda \to \infty$ *where*

$$\begin{aligned}
G(\lambda) &= \begin{pmatrix} \frac{1}{h_n} & \frac{th_n}{4\pi^{3/2}}(-1)^{n+1}(t\lambda)^{-\frac{3}{4}} & \frac{\mathrm{i}th_n}{4\pi^{3/2}}(t\lambda)^{-\frac{3}{4}}\\ \frac{\lambda^{-1}}{h_{n-1}} & \frac{th_{n-1}}{4\pi^{3/2}}(-1)^n(t\lambda)^{-\frac{1}{4}} & \frac{\mathrm{i}th_{n-1}}{4\pi^{3/2}}(t\lambda)^{-\frac{1}{4}}\\ \frac{\lambda^{-2}}{h_{n-2}} & \frac{th_{n-2}}{4\pi^{3/2}}(-1)^{n-1}(t\lambda)^{\frac{1}{4}} & \frac{\mathrm{i}th_{n-2}}{4\pi^{3/2}}(t\lambda)^{\frac{1}{4}} \end{pmatrix}\\
\Theta(\lambda) &= \mathrm{diag}\left(-\tfrac{1}{3}\lambda^3 - x\lambda + n\ln\lambda, \tfrac{2}{3}(t\lambda)^{3/2} - y(t\lambda)^{1/2} - \tfrac{n}{2}\ln(t\lambda)\right.\\
&\qquad \left. - \tfrac{2}{3}(t\lambda)^{3/2} + y(t\lambda)^{1/2} - \tfrac{n}{2}\ln(t\lambda)\right);
\end{aligned} \tag{37}$$

2. *Across the rays* $\arg\lambda = \frac{2\pi}{3}(j-1)$, $j = 1, 2, 3$, *oriented towards infinity,* $\Psi_n^{\mathrm{RH}}(\lambda)$ *has the jumps*

$$\Psi_n^{\mathrm{RH}+}(\lambda) = \Psi_n^{\mathrm{RH}-}(\lambda) S_j \qquad \arg\lambda = \tfrac{2\pi}{3}(j-1) \tag{38}$$

where plus and minus indicate the limiting values of $\Psi_n^{\mathrm{RH}}(\lambda)$ *on the jump contour from the left and from the right, respectively,*

$$S_1 = \begin{pmatrix} 1 & \left(g_1^{(2)} - g_0^{(2)}\right)\left(g_2^{(1)} - g_1^{(1)}\right) & \left(g_1^{(2)} - g_2^{(2)}\right)\left(g_1^{(1)} - g_2^{(1)}\right) \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

$$S_2 = \begin{pmatrix} 1 & \left(g_0^{(2)} - g_1^{(2)}\right)\left(g_2^{(1)} - g_0^{(1)}\right) & \left(g_0^{(2)} - g_2^{(2)}\right)\left(g_2^{(1)} - g_0^{(1)}\right) \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$S_3 = \begin{pmatrix} 1 & \left(g_1^{(2)} - g_2^{(2)}\right)\left(g_1^{(1)} - g_0^{(1)}\right) & \left(g_2^{(2)} - g_0^{(2)}\right)\left(g_1^{(1)} - g_0^{(1)}\right) \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}.$$

Across the ray $\arg\lambda = -\frac{\pi}{3}$ *oriented towards infinity, the jump condition holds*

$$\Psi_n^{\mathrm{RH}+}(\lambda) = \Psi_n^{\mathrm{RH}-}(\lambda)\Sigma \qquad \arg\lambda = -\frac{\pi}{3} \qquad \Sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{39}$$

The wavefunctions $\psi_n(\lambda)$, $\psi_{n-1}(\lambda)$ and $\psi_{n-2}(\lambda)$ are just entries of the first column of $\Psi_n^{\mathrm{RH}}(\lambda)$.

Remark 2. One can check the validity of the cyclic relation, $S_1 S_2 S_3 \Sigma = I$. It ensures the continuity of the RH problem above at the origin and reflects the fact that all solutions of an ODE with polynomial coefficients are entire functions.

5. Discussion

Assuming above the existence of bi-orthogonal polynomials for the cubic potentials, we constructed the fundamental matrix solutions Ψ, $\hat{\Psi}$ for equations (11) and (13), and, using these solutions, formulated matrix Riemann–Hilbert problems (37)–(39) and (22)–(24).

Alternatively, matrix equations (11) and (13) can be studied using the complex WKB method, see, e.g. [21]. In fact, asymptotic conditions (37) and (22) immediately come from the leading-order WKB asymptotic solutions to (11) and (13), respectively, while the jump matrices S_j in (38) and $\hat{S}_j$ in (23) are the special cases of the relevant general Stokes matrices. Actually, one finds S_j in (38) using the cyclic relation and assuming that the first column of the matrix solution to (11) has uniform asymptotics as $\lambda \to \infty$. Similarly, one finds $\hat{S}_j$ in (23) using the cyclic relation and assuming that the last column of the matrix solution to (13) does not affect the jumps of the remaining columns. However, the expressions for the nontrivial entries of the jump matrices in terms of the coefficients $g_i^{(j)}$ of (2) cannot be reproduced in this way.

The results on the Painlevé property and the solvability of the inverse monodromy problem obtained in [22–24] imply the solvability of the RH problems above for any given coefficients $g_i^{(j)}$ and generic values of the deformation parameters x, y, t.

Uniqueness of these solutions can be obtained in the usual way. For instance, consider RH problem 2. Introduce the matrix function $Y = G^{-1}\Psi^{\mathrm{RH}}\,\mathrm{e}^{-\Theta}$. It is straightforward that $\det Y$ is an entire function of λ. Moreover, $\det Y \equiv 1$ due to the Liouville theorem and normalization (37). Consider two solutions Ψ^{RH} and $\tilde{\Psi}^{\mathrm{RH}}$ of RH problem 2 and the respective functions Y and $\tilde{Y}$. The ratio $Z = Y\tilde{Y}^{-1}$ is an entire function of λ, moreover $Z \equiv I$ due to the Liouville theorem and normalization (37). Thus $\Psi^{\mathrm{RH}} \equiv \tilde{\Psi}^{\mathrm{RH}}$. Uniqueness of the solution to RH problem 1 is similar.

The above discussion of the unique solvability of the RH problems is independent from the fact of existence of the bi-orthogonal polynomials assumed in sections 3 and 4. Following [19] and using a determinant representation for the bi-orthogonal polynomials, it is possible to justify this assumption for arbitrary values of the deformation parameters x, y, t and some generic values of $g_i^{(j)}$.

We observe another opportunity to make sure that the set of bi-orthogonal polynomials for any $g_i^{(j)}$ and generic x, y, t is not empty. Namely, we conjecture that the *solvability of the RH problems for* Ψ^{RH}, $\hat{\Psi}^{\mathrm{RH}}$, Φ^{RH}, $\hat{\Phi}^{\mathrm{RH}}$ *leads to the existence of bi-orthogonal polynomials*. This assertion can be obtained as follows. The solution $\Psi^{\mathrm{RH}}(\lambda)$ of RH problem 2 contains in its first column a polynomial vector. The solution $\hat{\Psi}^{\mathrm{RH}}(\xi)$ of RH problem 1 gives rise to a matrix function $\hat{\Psi}(\xi)$ with the jump property (21). Thus the first two columns of $\hat{\Psi}(\xi)$ are entire functions, while the last column admits a Cauchy integral representation. Then the prescribed asymptotics at infinity of the latter yields the orthogonality of a combination of the first two columns to $\xi^m \exp(-W(\xi))$ for $m < n$. To complete the proof it is enough to observe that the first two columns of $\hat{\Psi}(\xi)$ are just Fourier–Laplace transforms of the first column in $\Psi^{\mathrm{RH}}(\lambda)$.

Finally, we note that the RH problems constructed above are useful for the study of n-large asymptotics of the bi-orthogonal polynomials. The necessary preparatory steps and anticipated results are announced in [25].

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References

[1] Plancherel M and Rotach W 1929 Sur les valeurs asymptotiques des poynomes d'Hermite $H_n(x) = (-1)^n \mathrm{e}^{x^2/2} d^n(\mathrm{e}^{-x^2/2})/\mathrm{d}x^n$ *Comment. Math. Helv.* **1** 227–54

[2] Szegö G 1939 *Orthogonal Polynomials* (Providence, RI: American Mathematical Society)

[3] Freud G 1976 On the coefficients in the recursion formulae of orthogonal polynomials *Proc. R. Ir. Acad.* A **76** 1–6

[4] Nevai P 1986 Géza Freud, orthogonal polynomials and Christoffel functions. A case study *J. Approx. Theory* **48** 3–167

[5] Sheen R -C 1987 Plancherel–Rotach-type asymptotics for orthogonal polynomials associated with $\exp(-x^6/6)$ *J. Approx. Theory* **50** 232–93

[6] Magnus A P 1995 Painlevé-type differential equations for the recurrence coefficients of semi-classical orthogonal polynomials *J. Comput. Appl. Math.* **57** 215–37

[7] Fokas A S, Its A R and Kitaev A V 1990 Isomonodromic approach in the theory of two-dimensional quantum gravity *Usp. Mat. Nauk* **45** 135–6

Fokas A S, Its A R and Kitaev A V 1991 Discrete Painlevé equations and their appearance in quantum gravity *Comment Math. Phys.* **142** 313–44

Fokas A S, Its A R and Kitaev A V 1992 The isomonodromy approach to matrix models in 2D quantum gravity *Commun. Math. Phys.* **147** 395–430

[8] Deift P A and Zhou X 1993 A steepest descent method for oscillatory Riemann–Hilbert problems. Asymptotics for the MKdV equation *Ann. Math.* **137** 295–368

[9] Bleher P M and Its A R 1999 Semiclassical asymptotics of orthogonal polynomials, Riemann–Hilbert problem, and universality in the matrix model *Ann. Math.* **150** 185–266

Bleher P M and Its A R 2002 Double scaling limit in the random matrix model: Riemann–Hilbert approach *Preprint* math-ph/0201003

[10] Deift P, Kriecherbauer T, McLaughlin K T-R, Venakides S and Zhou X 1999 Uniform asymptotics for polynomials orthogonal with respect to varying exponential weights and applications to universality questions in random matrix theory *Commun. Pure Appl. Math.* **52** 1335–425

Deift P, Kriecherbauer T, McLaughlin K T-R, Venakides S and Zhou X 1999 Strong asymptotics of orthogonal polynomials with respect to exponential weights *Commun. Pure Appl. Math.* **52** 1491–552

[11] Baik J, Deift P and Johansson K 1999 On the distribution of the length of the longest increasing subsequence of random permutations *J. Am. Math. Soc.* **12** 1119–78 (*Preprint* math.CO/9810105)

[12] Adler M and van Moerbeke P 2000 Generalized orthogonal polynomials, discrete KP and Riemann–Hilbert problems *Preprint* nlin.SI/0009002

[13] van Moerbeke P 2001 *Integrable Lattices: Random Matrices and Random Permutations (MSRI Publications No 40)* (Cambridge: Cambridge University Press) (*Preprint* math.CO/0010135)

[14] Ercolani N M and McLaughlin K T-R 2001 Asymptotics and integrable structures for biorthogonal polynomials associated to a random two-matrix model *Physica* D **152–153** 232–68

[15] Bertola M, Eynard B and Harnad J 2001 Duality, biorthogonal polynomials and multi-matrix models *Preprint* nlin.SI/0108049

[16] Zinn-Justin P 2002 HCIZ integral and 2D Toda lattice hierarchy *Preprint* math-ph/0202045

[17] Bertola M 2002 Bilinear semi-classical moment functionals and their integral representation *Preprint* math.CA/0205160

[18] Kapaev A A 2002 The Riemann–Hilbert problem for the bi-orthogonal polynomials *Preprint* nlin.SI/0207036

[19] Bertola M, Eynard B and Harnad J 2002 Differential systems appearing in 2-matrix models and the associated Riemann–Hilbert problem *Preprint* nlin.SI/0208002

[20] Its A R and Kapaev A A 2002 The nonlinear steepest descent approach to the asymptotics of the second Painleve transcendent in the complex domain *MathPhys Odyssey 2001, Integrable Models and Beyond* (in Honour of Barry M McCoy) ed M Kashiwara and T Miwa *(Progress in Mathematical Physics vol 23)* (Boston, MA: Birkhäuser) pp 273–311 (*Preprint* nlin.SI/0108054)

[21] Wasow W 1965 *Asymptotic Expansions for Ordinary Differential Equations* (New York: Interscience/Wiley)

[22] Miwa T 1981 Painlevé property of monodromy preserving equations and the analyticity of τ-functions *Publ. Res. Inst. Math. Sci.* **17** 703–21

[23] Malgrange B 1991 *Differential Equations with Polynomial Coefficients (Progress in Mathematics vol 96)* (Boston, MA: Birkhäuser)

[24] Bolibrukh A, Its A R and Kapaev A A On the Riemann–Hilbert–Birkhoff inverse monodromy problem and the Painlevé equations In preparation

[25] Kapaev A A 2002 Monodromy approach to the scaling limits in the isomonodromy systems *Preprint* nlin.SI/0211022